

Space-time deformations as extended conformal transformations

S. Capozziello^{a,b} and C. Stornaiolo^{b*}

^a *Dipartimento di Scienze Fisiche, Università di Napoli "Federico II",
and ^bINFN, Sez. di Napoli, Compl. Univ. di Monte S. Angelo,
Edificio G, Via Cinthia, I-80126 - Napoli, Italy*

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A definition of space-time metric deformations on an n -dimensional manifold is given. We show that such deformations can be regarded as extended conformal transformations. In particular, their features can be related to the perturbation theory giving a natural picture by which gravitational waves are described by small deformations of the metric. As further result, deformations can be related to approximate Killing vectors (approximate symmetries) by which it is possible to parameterize the deformed region of a given manifold. The perspectives and some possible physical applications of such an approach are discussed.

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I. INTRODUCTION

The issue to consider a general way to deform the space-time metrics is not new. It has been posed in different ways and is related to several physical problems ranging from the spontaneous symmetry breaking of unification theories up to gravitational waves, considered as space-time perturbations. In cosmology, for example, one faces the problem to describe an observationally lumpy universe at small scales which becomes isotropic and homogeneous at very large scales according to the Cosmological Principle. In this context, it is crucial to find a way to connect background and locally perturbed metrics [1]. For example, McVittie [2] considered a metric which behaves as a Schwarzschild one at short ranges and as a Friedman-Lemaitre-Robertson-Walker metric at very large scales. Gautreau [3] calculated the metric generated by a Schwarzschild mass embedded in a Friedman cosmological fluid trying to address the same problem. On the other hand, the post-newtonian parameterization, as a standard, can be considered as a deformation of a background, asymptotically flat Minkowski metric.

In general, the deformation problem has been explicitly posed by Coll and collaborators [4, 5, 10] who conjectured the possibility to obtain any metric from the deformation of a space-time with constant curvature. The problem was solved only for 3-dimensional spaces but a straightforward extension should be to achieve the same result for space-times of any dimension.

In principle, new exact solutions of the Einstein field equations can be obtained by studying perturbations. In particular, dealing with perturbations as Lorentz matrices of scalar fields Φ^A_C reveals particularly useful. Firstly they transform as scalars with respect the coordinate transformations. Secondly, they are dimensionless and, in each point, the matrix Φ^A_C behaves as the element of a group. As we shall see below, such an approach can be related to the conformal transformations giving an "extended" interpretation and a straightforward physical meaning of them (see [7, 8] and references therein for a comprehensive review). Furthermore scalar fields related to space-time deformations have a straightforward physical interpretation which could contribute to explain several fundamental issues as the Higgs mechanism in unification theories, the inflation in cosmology and other pictures where scalar fields play a fundamental role in dynamics.

In this paper, we are going to discuss the properties of the deforming matrices Φ^A_C and we will derive, from the Einstein equations, the field equations for them, showing how them can parameterize the deformed metrics, according to the boundary and initial conditions and to the energy-momentum tensor.

The layout of the paper is the following. In Sec.II, we define the space-time perturbations in the framework of the metric formalism giving the notion of first and second deformation matrices. Sec.III is devoted to the main properties of deformations. In particular, we discuss how deformation matrices can be split in their trace, traceless and skew parts. We derive the contributions of deformation to the geodesic equation and, starting from the curvature Riemann tensor, the general equation of deformations. In Sec.IV we discuss the notion of linear perturbations under the standard of deformations. In particular, we recast the equation of gravitational waves and the transverse traceless

*Electronic address: capozziello@na.infn.it, stornaiolo@na.infn.it

gauge under the standard of deformations. Sec.V is devoted to discuss the action of deformations on the Killing vectors. The result consists in achieving a notion of approximate symmetry. Discussion and conclusions are given in Sec.VI. In Appendix, we discuss in details how deformations act on affine connections.

II. GENERALITIES ON SPACE-TIME DEFORMATIONS

In order to start our considerations, let us take into account a metric \mathbf{g} on a space-time manifold \mathcal{M} . Such a metric is assumed to be an exact solution of the Einstein field equations. We can decompose it by a co-tetrad field $\omega^A(x)$

$$\mathbf{g} = \eta_{AB} \omega^A \omega^B. \quad (1)$$

Let us define now a new tetrad field $\tilde{\omega} = \Phi^A_C(x) \omega^C$, with $\Phi^A_C(x)$ a matrix of scalar fields. Finally we introduce a space-time $\tilde{\mathcal{M}}$ with the metric \tilde{g} defined in the following way

$$\tilde{\mathbf{g}} = \eta_{AB} \Phi^A_C \Phi^B_D \omega^C \omega^D = \gamma_{CD}(x) \omega^C \omega^D, \quad (2)$$

where also $\gamma_{CD}(x)$ is a matrix of fields which are scalars with respect to the coordinate transformations.

If $\Phi^A_C(x)$ is a Lorentz matrix in any point of \mathcal{M} , then

$$\tilde{g} \equiv g \quad (3)$$

otherwise we say that \tilde{g} is a deformation of g and $\tilde{\mathcal{M}}$ is a deformed \mathcal{M} . If all the functions of $\Phi^A_C(x)$ are continuous, then there is a *one - to - one* correspondence between the points of \mathcal{M} and the points of $\tilde{\mathcal{M}}$.

In particular, if ξ is a Killing vector for g on \mathcal{M} , the corresponding vector $\tilde{\xi}$ on $\tilde{\mathcal{M}}$ could not necessarily be a Killing vector.

A particular subset of these deformation matrices is given by

$$\Phi^A_C(x) = \Omega(x) \delta^A_C. \quad (4)$$

which define conformal transformations of the metric,

$$\tilde{g} = \Omega^2(x) g. \quad (5)$$

In this sense, the deformations defined by Eq. (2) can be regarded as a generalization of the conformal transformations.

We call the matrices $\Phi^A_C(x)$ *first deformation matrices*, while we can refer to

$$\gamma_{CD}(x) = \eta_{AB} \Phi^A_C(x) \Phi^B_D(x). \quad (6)$$

as the *second deformation matrices*, which, as seen above, are also matrices of scalar fields. They generalize the Minkowski matrix η_{AB} with constant elements in the definition of the metric. A further restriction on the matrices Φ^A_C comes from the theorem proved by Riemann by which an n -dimensional metric has $n(n-1)/2$ degrees of freedom (see [5] for details). With this definitions in mind, let us consider the main properties of deforming matrices.

III. PROPERTIES OF DEFORMING MATRICES

Let us take into account a four dimensional space-time with Lorentzian signature. A family of matrices $\Phi^A_C(x)$ such that

$$\Phi^A_C(x) \in GL(4) \forall x, \quad (7)$$

are defined on such a space-time.

These functions are not necessarily continuous and can connect space-times with different topologies. A singular scalar field introduces a deformed manifold $\tilde{\mathcal{M}}$ with a space-time singularity.

As it is well known, the Lorentz matrices Λ^A_C leave the Minkowski metric invariant and then

$$\mathbf{g} = \eta_{EF} \Lambda^E_A \Lambda^F_B \Phi^A_C \Phi^B_D \omega^C \omega^D = \eta_{AB} \Phi^A_C \Phi^B_D \omega^C \omega^D. \quad (8)$$

It follows that Φ^A_C give rise to right cosets of the Lorentz group, i.e. they are the elements of the quotient group $GL(4, \mathbf{R})/SO(3, 1)$. On the other hand, a right-multiplication of Φ^A_C by a Lorentz matrix induces a different deformation matrix.

The inverse deformed metric is

$$\tilde{g}^{ab} = \eta^{CD} \Phi^{-1A}_C \Phi^{-1B}_D e^a_A e^b_B \quad (9)$$

where $\Phi^{-1A}_C \Phi^C_B = \Phi^A_C \Phi^{-1C}_B = \delta^A_B$.

Let us decompose now the matrix $\Phi_{AB} = \eta_{AC} \Phi^C_B$ in its symmetric and antisymmetric parts

$$\Phi_{AB} = \Phi_{(AB)} + \Phi_{[AB]} = \Omega \eta_{AB} + \Theta_{AB} + \varphi_{AB} \quad (10)$$

where $\Omega = \Phi^A_A$, Θ_{AB} is the traceless symmetric part and φ_{AB} is the skew symmetric part of the first deformation matrix respectively. Then standard conformal transformations are nothing else but deformations with $\Theta_{AB} = \varphi_{AB} = 0$ [9].

Finding the inverse matrix Φ^{-1A}_C in terms of Ω , Θ_{AB} and φ_{AB} is not immediate, but as above, it can be split in the three terms

$$\Phi^{-1A}_C = \alpha \delta^A_C + \Psi^A_C + \Sigma^A_C \quad (11)$$

where α , Ψ^A_C and Σ^A_C are respectively the trace, the traceless symmetric part and the antisymmetric part of the inverse deformation matrix. The second deformation matrix, from the above decomposition, takes the form

$$\gamma_{AB} = \eta_{CD} (\Omega \delta^C_A + \Theta^C_A + \varphi^C_A) (\Omega \delta^D_B + \Theta^D_B + \varphi^D_B) \quad (12)$$

and then

$$\begin{aligned} \gamma_{AB} = & \Omega^2 \eta_{AB} + 2\Omega \Theta_{AB} + \eta_{CD} \Theta^C_A \Theta^D_B + \eta_{CD} (\Theta^C_A \varphi^D_B \\ & + \varphi^C_A \Theta^D_B) + \eta_{CD} \varphi^C_A \varphi^D_B. \end{aligned} \quad (13)$$

In general, the deformed metric can be split as

$$\tilde{g}_{ab} = \Omega^2 g_{ab} + \gamma_{ab} \quad (14)$$

where

$$\begin{aligned} \gamma_{ab} = & (2\Omega \Theta_{AB} + \eta_{CD} \Theta^C_A \Theta^D_B + \eta_{CD} (\Theta^C_A \varphi^D_B + \varphi^C_A \Theta^D_B) \\ & + \eta_{CD} \varphi^C_A \varphi^D_B) \omega^A_a \omega^B_b \end{aligned} \quad (15)$$

In particular, if $\Theta_{AB} = 0$, the deformed metric simplifies to

$$\tilde{g}_{ab} = \Omega^2 g_{ab} + \eta_{CD} \varphi^C_A \varphi^D_B \omega^A_a \omega^B_b \quad (16)$$

and, if $\Omega = 1$, the deformation of a metric consists in adding to the background metric a tensor γ_{ab} . We have to remember that all these quantities are not independent as, by the theorem mentioned in [5], they have to form at most six independent functions in a four dimensional space-time.

Similarly the contravariant deformed metric can be always decomposed in the following way

$$\tilde{g}^{ab} = \alpha^2 g^{ab} + \lambda^{ab} \quad (17)$$

Let us find the relation between γ_{ab} and λ^{ab} . By using $\widetilde{g_{ab} g^{bc}} = \delta^c_a$, we obtain

$$\alpha^2 \Omega^2 \delta^c_a + \alpha^2 \gamma^c_a + \Omega^2 \lambda^c_a + \gamma_{ab} \lambda^{bc} = \delta^c_a \quad (18)$$

if the deformations are conformal transformations, we have $\alpha = \Omega^{-1}$, so assuming such a condition, one obtain the following matrix equation

$$\alpha^2 \gamma^c_a + \Omega^2 \lambda^c_a + \gamma_{ab} \lambda^{bc} = 0, \quad (19)$$

and

$$(\delta_a^b + \Omega^{-2}\gamma_a^b)\lambda_b^c = -\Omega^{-4}\gamma_a^c \quad (20)$$

and finally

$$\lambda_b^c = -\Omega^{-4}(\delta + \Omega^{-2}\gamma)^{-1}{}_b^a \gamma_a^c \quad (21)$$

where $(\delta + \Omega^{-2}\gamma)^{-1}$ is the inverse tensor of $(\delta_a^b + \Omega^{-2}\gamma_a^b)$.

To each matrix $\Phi^A{}_B$, we can associate a (1,1)-tensor $\phi^a{}_b$ defined by

$$\phi^a{}_b = \Phi^A{}_B \omega_b^B e_A^a \quad (22)$$

such that

$$\tilde{g}_{ab} = g_{cd} \phi^c{}_a \phi^d{}_b \quad (23)$$

which can be decomposed as in Eq.(16). Vice-versa from a (1,1)-tensor $\phi^a{}_b$, we can define a matrix of scalar fields as

$$\phi^A{}_B = \phi^a{}_b \omega_a^A e_B^b. \quad (24)$$

The Levi Civita connection corresponding to the metric (14) is related to the original connection by the relation (see the Appendix for details)

$$\tilde{\Gamma}_{ab}^c = \Gamma_{ab}^c + C_{ab}^c \quad (25)$$

(see [9]), where

$$C_{ab}^c = 2\tilde{g}^{cd}g_{d(a}\nabla_{b)}\Omega - g_{ab}\tilde{g}^{cd}\nabla_d\Omega + \frac{1}{2}\tilde{g}^{cd}(\nabla_a\gamma_{db} + \nabla_b\gamma_{ad} - \nabla_d\gamma_{ab}). \quad (26)$$

Therefore, in a deformed space-time, the connection deformation acts like a force that deviates the test particles from the geodesic motion in the unperturbed space-time. As a matter of fact the geodesic equation for the deformed space-time

$$\frac{d^2x^c}{d\lambda^2} + \tilde{\Gamma}_{ab}^c \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} = 0 \quad (27)$$

becomes

$$\frac{d^2x^c}{d\lambda^2} + \Gamma_{ab}^c \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda} = -C_{ab}^c \frac{dx^a}{d\lambda} \frac{dx^b}{d\lambda}. \quad (28)$$

The deformed Riemann curvature tensor is then

$$\tilde{R}_{abc}{}^d = R_{abc}{}^d + \nabla_b C_{ac}^d - \nabla_a C_{bc}^d + C_{ac}^e C_{be}^d - C_{bc}^e C_{ae}^d, \quad (29)$$

while the deformed Ricci tensor obtained by contraction is

$$\tilde{R}_{ab} = R_{ab} + \nabla_d C_{ab}^d - \nabla_a C_{db}^d + C_{ab}^e C_{de}^d - C_{db}^e C_{ae}^d \quad (30)$$

and the curvature scalar

$$\tilde{R} = \tilde{g}^{ab}\tilde{R}_{ab} = \tilde{g}^{ab}R_{ab} + \tilde{g}^{ab}[\nabla_d C_{ab}^d - \nabla_a C_{db}^d + C_{ab}^e C_{de}^d - C_{db}^e C_{ae}^d] \quad (31)$$

From the above curvature quantities, we obtain finally the equations for the deformations. In the vacuum case, we simply have

$$\tilde{R}_{ab} = R_{ab} + \nabla_d C_{ab}^d - \nabla_a C_{db}^d + C_{ab}^e C_{de}^d - C_{db}^e C_{ae}^d = 0 \quad (32)$$

where R_{ab} must be regarded as a known function. In presence of matter, we consider the equation

$$R_{ab} + \nabla_d C^d_{ab} - \nabla_a C^d_{db} + C^e_{ab} C^d_{de} - C^e_{db} C^d_{ae} = \tilde{T}_{ab} - \frac{1}{2} \tilde{g}_{ab} \tilde{T} \quad (33)$$

we are assuming, for the sake of simplicity $8\pi G = c = 1$. This last equation can be improved by considering the Einstein field equations

$$R_{ab} = T_{ab} - \frac{1}{2} g_{ab} T \quad (34)$$

and then

$$\nabla_d C^d_{ab} - \nabla_a C^d_{db} + C^e_{ab} C^d_{de} - C^e_{db} C^d_{ae} = \tilde{T}_{ab} - \frac{1}{2} \tilde{g}_{ab} \tilde{T} - \left(T_{ab} - \frac{1}{2} g_{ab} T \right) \quad (35)$$

is the most general equation for deformations.

IV. METRIC DEFORMATIONS AS PERTURBATIONS AND GRAVITATIONAL WAVES

Metric deformations can be used to describe perturbations. To this aim we can simply consider the deformations

$$\Phi^A_B = \delta^A_B + \varphi^A_B \quad (36)$$

with

$$|\varphi^A_B| \ll 1, \quad (37)$$

together with their derivatives

$$|\partial \varphi^A_B| \ll 1. \quad (38)$$

With this approximation, immediately we find the inverse relation

$$(\Phi^{-1})^A_B \simeq \delta^A_B - \varphi^A_B. \quad (39)$$

As a remarkable example, we have that gravitational waves are generally described, in linear approximation, as perturbations of the Minkowski metric

$$g_{ab} = \eta_{ab} + \gamma_{ab}. \quad (40)$$

In our case, we can extend in a covariant way such an approximation. If φ_{AB} is an antisymmetric matrix, we have

$$\tilde{g}_{ab} = g_{ab} + \gamma_{ab} \quad (41)$$

where the first order terms in φ^A_B vanish and γ_{ab} is of second order

$$\gamma_{ab} = \eta_{AB} \varphi^A_C \varphi^B_D \omega^C_a \omega^D_b. \quad (42)$$

Consequently

$$\tilde{g}^{ab} = g^{ab} + \gamma^{ab} \quad (43)$$

where

$$\gamma^{ab} = \eta^{AB} (\varphi^{-1})^C_A (\varphi^{-1})^D_B e_C^a e_D^b. \quad (44)$$

Let us consider the background metric g_{ab} , solution of the Einstein equations in the vacuum

$$R_{ab} = 0. \quad (45)$$

We obtain the equation of perturbations considering only the linear terms in Eq.(32) and neglecting the contributions of quadratic terms. We find

$$\tilde{R}_{ab} = \nabla_d C^d_{ab} - \nabla_a C^d_{db} = 0, \quad (46)$$

and, by the explicit form of C^d_{ab} , this equation becomes

$$(\nabla_d \nabla_a \gamma^d_b + \nabla_d \nabla_b \gamma^d_a - \nabla_d \nabla^d \gamma_{ab}) - (\nabla_a \nabla_d \gamma^d_b + \nabla_a \nabla_b \gamma^d_d - \nabla_a \nabla^d \gamma^d_b) = 0. \quad (47)$$

Imposing the transverse traceless gauge on γ_{ab} , i.e. the standard gauge conditions

$$\nabla^a \gamma_{ab} = 0 \quad (48)$$

and

$$\gamma = \gamma^a_a = 0 \quad (49)$$

Eq.(47) reduces to

$$\nabla_b \nabla^b \gamma_{ac} - 2R^b_{ac} \gamma_{bd} = 0, \quad (50)$$

see also [9]. In our context, this equation is a linearized equation for deformations and it is straightforward to consider perturbations and, in particular, gravitational waves, as small deformations of the metric. This result can be immediately translated into the above scalar field matrix equations. Note that such an equation can be applied to the conformal part of the deformation, when the general decomposition is considered.

As an example, let us take into account the deformation matrix equations applied to the Minkowski metric, when the deformation matrix assumes the form (36). In this case, the equations (47), become ordinary wave equations for γ_{ab} . Considering the deformation matrices, these equations become, for a tetrad field of constant vectors,

$$\partial^d \partial_d \varphi^C_A \varphi_{CB} + 2 \partial_d \varphi^C_A \partial^d \varphi_{CB} + \varphi^C_A \partial^d \partial_d \varphi_{CB} = 0. \quad (51)$$

The above gauge conditions are now

$$\varphi_{AB} \varphi^{BA} = 0 \quad (52)$$

and

$$e^d_D [\partial_d \varphi_{CA} \varphi^C_B + \varphi_{CA} \partial_d \varphi^C_B] = 0. \quad (53)$$

This result shows that the gravitational waves can be fully recovered starting from the scalar fields which describe the deformations of the metric. In other words, such scalar fields can assume the meaning of gravitational wave modes.

V. APPROXIMATE KILLING VECTORS

Another important issue which can be addressed starting from space-time deformations is related to the symmetries. In particular, they assume a fundamental role in describing when a symmetry is preserved or broken under the action of a given field. In General Relativity, the Killing vectors are always related to the presence of given space-time symmetries [9].

Let us take an exact solution of the Einstein equations, which satisfies the Killing equation

$$(L_\xi g)_{ab} = 0 \quad (54)$$

where ξ , being the generator of an infinitesimal coordinate transformation, is a Killing vector. If we take a deformation of the metric with the scalar matrix

$$\Phi^A_B = \delta^A_B + \varphi^A_B \quad (55)$$

with

$$|\varphi^A_B| \ll 1, \quad (56)$$

and

$$(L_\xi \tilde{g})_{ab} \neq 0, \quad (57)$$

being

$$(L_\xi e^A)_a = 0, \quad (58)$$

we have

$$(L_\xi \varphi)^A_B = \xi^a \partial_a \varphi^A_B \neq 0. \quad (59)$$

If there is some region \mathcal{D} of the deformed space-time $\mathcal{M}_{deformed}$ where

$$|(L_\xi \varphi)^A_B| \ll 1 \quad (60)$$

we say that ξ is an *approximate Killing vector* on \mathcal{D} . In other words, these approximate Killing vectors allow to "control" the space-time symmetries under the action of a given deformation.

VI. DISCUSSION AND CONCLUSIONS

In this paper, we have proposed a novel definition of space-time metric deformations parameterizing them in terms of scalar field matrices. The main result is that deformations can be described as extended conformal transformations. This fact gives a straightforward physical interpretation of conformal transformations: conformally related metrics can be seen as the "background" and the "perturbed" metrics. In other words, the relations between the Jordan frame and the Einstein frame can be directly interpreted through the action of the deformation matrices contributing to solve the issue of what the true physical frame is [7, 8].

Besides, space-time metric deformations can be immediately recast in terms of perturbation theory allowing a completely covariant approach to the problem of gravitational waves.

Results related to those presented here has been proposed in [4, 5]. There it is shown that any metric in a three dimensional manifold can be decomposed in the form

$$\tilde{g}_{ab} = \sigma(x) h_{ab} + \epsilon s_a s_b \quad (61)$$

where h_{ab} is a metric with constant curvature, $\sigma(x)$ is a scalar function, s_a is a three-vector and $\epsilon = \pm 1$. A relation has to be imposed between σ and s_a and then the metric can be defined, at most, by three independent functions.

In a subsequent paper [6], Llosa and Soler showed that (61) can be generalized to arbitrary dimensions by the form

$$\tilde{g}_{ab} = \lambda(x) g_{ab} + \mu(x) F_{ac} g^{cd} F_{db} \quad (62)$$

where g_{ab} is a constant curvature metric, F_{ab} is a two-form, $\lambda(x)$ and $\mu(x)$ are two scalar functions. These results are fully recovered and generalized from our approach as soon as the deformation of a constant metric is considered and suitable conditions on the tensor Θ_{AB} are imposed.

In general, we have shown that, when we turn to the tensor formalism, we can work with arbitrary metrics and arbitrary deforming γ_{ab} tensors. In principle, by arbitrary deformation matrices, not necessarily real, we can pass from a given metric to any other metric. As an example, a noteworthy result has been achieved by Newman and Janis [11]: They showed that, through a complex coordinate transformation, it is always possible to achieve a Kerr metric from a Schwarzschild one. In our language, this means that a space-time deformation allows to pass from a spherical symmetry to a cylindrical one. Furthermore, it has been shown [12, 13] that three dimensional black hole solutions can be found by identifying 3-dimensional anti-de Sitter space on which acts a discrete subgroup of $SO(2, 2)$.

In all these examples, the transformations which lead to the results are considered as "coordinate transformations". We think that this definition is a little bit misleading since one does not covariantly perform the same transformations on *all* the tensors defined on the manifold. On the other hand, our definition of metric deformations and deformed manifolds can be straightforwardly related to the standard notion of perturbations since, in principle, it works on a given region \mathcal{D} of the deformed space-time (see, for example, [14, 15]).

VII. APPENDIX

We can calculate the modified connection $\hat{\Gamma}_{ab}^c$ in many alternative ways. Let us introduce the tetrad e_A and cotetrad ω^B satisfying the orthogonality relation

$$i_{e_A}\omega^B = \delta_A^B \quad (63)$$

and the non-integrability condition (anholonomy)

$$d\omega^A = \frac{1}{2}\Omega_{BC}^A\omega^B \wedge \omega^C. \quad (64)$$

The corresponding connection is

$$\Gamma_{BC}^A = \frac{1}{2} \left(\Omega_{BC}^A - \eta^{AA'}\eta_{BB'}\Omega_{A'C}^{B'} - \eta^{AA'}\eta_{CC'}\Omega_{A'B}^{C'} \right) \quad (65)$$

If we deform the metric as in (2), we have two alternative ways to write this expression: either writing the “deformation” of the metric in the space of tetrads or “deforming” the tetrad field as in the following expression

$$\hat{g} = \eta_{AB}\Phi^A{}_C\Phi^B{}_D\omega^C\omega^D = \gamma_{AB}\omega^A\omega^B = \eta_{AB}\hat{\omega}^A\hat{\omega}^B. \quad (66)$$

In the first case, the contribution of the Christoffel symbols, constructed by the metric γ_{AB} , appears

$$\begin{aligned} \hat{\Gamma}_{BC}^A &= \frac{1}{2} \left(\Omega_{BC}^A - \gamma^{AA'}\gamma_{BB'}\Omega_{A'C}^{B'} - \gamma^{AA'}\gamma_{CC'}\Omega_{A'B}^{C'} \right) \\ &\quad + \frac{1}{2}\gamma^{AA'}(i_{e_C}d\gamma_{BA'} - i_{e_B}d\gamma_{CA'} - i_{e_{A'}}d\gamma_{BC}) \end{aligned} \quad (67)$$

In the second case, using (64), we can define the new anholonomy objects \hat{C}_{BC}^A .

$$d\hat{\omega}^A = \frac{1}{2}\hat{\Omega}_{BC}^A\hat{\omega}^B \wedge \hat{\omega}^C. \quad (68)$$

After some calculations, we have

$$\hat{\Omega}_{BC}^A = \Phi^A{}_E\Phi^{-1D}{}_B\Phi^{-1F}{}_C\Omega_{DF}^E + 2\Phi^A{}_Fe_G^a \left(\Phi^{-1G}{}_{[B}\partial_a\Phi^{-1F}{}_{C]} \right) \quad (69)$$

As we are assuming a constant metric in tetradic space, the deformed connection is

$$\hat{\Gamma}_{BC}^A = \frac{1}{2} \left(\hat{\Omega}_{BC}^A - \eta^{AA'}\eta_{BB'}\hat{\Omega}_{A'C}^{B'} - \eta^{AA'}\eta_{CC'}\hat{\Omega}_{A'B}^{C'} \right). \quad (70)$$

Substituting (69) in (70), the final expression of $\hat{\Gamma}_{BC}^A$, as a function of Ω_{BC}^A , $\Phi^A{}_B$, $\Phi^{-1D}{}_C$ and e_G^a is

$$\hat{\Gamma}_{ABC} = \Delta_{ABC}^{DEF} \left[\frac{1}{2}\eta_{DG}\Phi^G{}_{G'}\Phi^{-1E'}{}_E\Phi^{-1F'}{}_F\Omega_{E'F'}^{G'} + \eta_{DK}\Phi^K{}_He_G^a\Phi^{-1G}{}_{[E}\partial_{|a|}\Phi^{-1H}{}_{F]} \right] \quad (71)$$

where

$$\Delta_{ABC}^{DEF} = \delta_A^D\delta_C^E\delta_B^F - \delta_B^D\delta_C^E\delta_A^F + \delta_C^D\delta_A^E\delta_B^F. \quad (72)$$

- [1] G. F. R. Ellis, *General Relativity and Gravitation*, GR10 Conf. Rep. Ed. B. Bertotti (Dordrecht: Reidel), p. 215 (1984).
- [2] G. C. McVittie, *The Mass-Particle in an Expanding Universe*, MNRAS **93**, p.325-3 (1933).
- [3] R. Gautreau, *Embedding a Schwarzschild mass into cosmology*, Phys. Rev. D **29**, 198-206 (1984).
- [4] B. Coll, *A universal law of gravitational deformation for general relativity*, Proc. of the Spanish Relativistic Meeting, EREs, Salamanca Spain (1998).

- [5] B. Coll, J. Llosa and D. Soler, *Three-dimensional metrics as deformations of a constant curvature metric*, Gen. Rel. Grav. **34** 269 (2002).
- [6] J. Llosa and D. Soler, *On the degrees of freedom of a semi-Riemannian metric*, Class. Quant. Grav. **22**, 893 (2005)
- [7] V. Faraoni, *Cosmology in Scalar-Tensor Gravity*, Kluwer Academic, Dordrecht (2004).
- [8] G. Allemandi, M. Capone, S. Capozziello, M. Francaviglia, *Conformal aspects of Palatini approach in Extended Theories of Gravity*, Gen. Rel. Grav. **38**, 33 (2006).
- [9] R. M. Wald, *General Relativity*, The University of Chicago Press, (1984).
- [10] J. Llosa and D. Soler, *Reference frames and rigid motions in relativity*, Class. Quant. Grav. **21** 3067 (2004).
- [11] E. T. Newman and A. I. Janis, *Note on the Kerr spinning particle metric*, J. Math. Phys. **6**, 915 (1965).
- [12] M. Banados, C. Teitelboim and J. Zanelli, *The Black hole in three-dimensional space-time*, Phys. Rev. Lett. **69**, 1849 (1992).
- [13] M. Banados, M. Henneaux, C. Teitelboim and J. Zanelli, *Geometry of the (2+1) black hole*, Phys. Rev. D **48** 1506, (1993)
- [14] J. M. Bardeen, *Gauge Invariant Cosmological Perturbations*, Phys. Rev. D **22**, 1882 (1980).
- [15] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, *Theory of cosmological perturbations. Part 1. Classical perturbations. Part 2. Quantum theory of perturbations. Part 3. Extensions*, Phys. Rept. **215**, 203 (1992).